Polyhedral Convexity and the Existence of Approximate Equilibria in Discontinuous Games*

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Abstract

Radzik (1991) showed that two-player games on compact intervals of the real line have $\varepsilon-$ equilibria for all $\varepsilon > 0$, provided that payoff functions are upper semicontinuous and strongly quasi-concave. In an attempt to generalize this theorem, Ziad (1997) stated that the same is true for n-player games on compact, convex subsets of $\mathbb{R}^m$, $m \geq 1$ provided that we strengthen the upper semicontinuity condition.

We show that:

1. the action spaces need to be polyhedral in order for Ziad's approach to work,
2. Ziad's strong upper semicontinuity condition is equivalent to some form of quasi-polyhedral concavity of players' value functions in simple games, and
3. Radzik's Theorem is a corollary of (the corrected) Ziad's result.

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1 Introduction

The standard approach to prove the existence of Nash equilibria in a normal form game is to show that the best-reply correspondence satisfies the conditions needed to apply a fixed point theorem (see, e.g., Nash (1950) and Glicksberg (1952)). Alternatively, one can show that the best-reply correspondence satisfies the conditions needed to use a continuous selection theorem, and then apply a fixed point theorem to this selection.

The latter approach was successfully pursued by Radzik (1991). There, he considered upper semicontinuous and strongly quasi-concave two-player games played on compact intervals on the real line and showed that for all $\varepsilon > 0$, the ε – best-reply correspondence is lower hemicontinuous with closed, convex values. Hence, it follows from Michael's selection theorem and Brouwer's fixed point theorem that such a game has an ε – equilibrium for all $\varepsilon > 0$.

In an attempt to generalize Radzik's theorem, Ziad (1997) claimed that the same approach could be used in n – person normal form games played on compact, convex subsets of $\mathbb{R}^m$, $m \geq 1$. All that seemed to be needed was a new condition, named i – upper semicontinuity, which appeared to be a strengthening of the upper semicontinuity of players' payoff functions.

In contrast to what was stated in Ziad (1997), we show that the action spaces need to be polyhedral in order for the $\varepsilon -$ best reply to be lower hemicontinuous. Furthermore, if players' payoff functions are quasi-concave on the joint action space, we show that Ziad's $i -$ upper semicontinuity corresponds to quasi-polyhedral concavity of players' value functions in simple games.[1] Hence, we conclude that polyhedral convexity is a key property in extending Radzik's approach from two-player games on the real line to $n -$ person games on higher dimensional euclidian spaces.

The relevance of our results can be understood as follows. The problem of existence of Nash equilibria is well understood in continuous, quasi-concave normal form games.[2] Clearly, a function is continuous if and only if it is upper and lower semicontinuous, and recent research has attempted to generalize such an existence theorem by weakening some or both semicontinuity assumptions.

Once one drops continuity, two questions arise, both related with the existence problem. One is: under what conditions does there exist an $\varepsilon -$

[1] This result holds with a weaker quasi-concavity requirement on players' payoff functions.

[2] These are games with convex, compact action spaces and continuous payoff functions that are quasi-concave in the owner's action.

equilibrium for all $\varepsilon > 0$? A second then becomes: when is the limit of a sequence of ε – equilibria, with ε converging to zero, a Nash equilibrium?

The result of Radzik (1991) answers the first question for two-player games on a square and shows that the lower semicontinuity can be dispensed with altogether once the quasi-concavity is strengthened. Our results, combined with the main result of Ziad (1997), show that the same extends to n – player games played on polytopes of $\mathbb{R}^m$, $m \geq 1$. In fact, every game with upper semicontinuous payoff functions and quasi-polyhedral concave value functions has an ε – equilibrium for all $\varepsilon > 0$.

2 Normal form games

A *normal form game* G consists of a finite set of players $N = \{1, \ldots, n\}$, and, for all players $i \in N$, a pure strategy set X_i, represented by a compact subset of $\mathbb{R}^m$, and a bounded payoff function $u_i : X \to \mathbb{R}$, where $X = \times_{i \in N} X_i$.

For all $y \in \mathbb{R}$, $|y|$ denotes the absolute value of y. Let $i \in N$. For all $x \in X_i$, let $||x|| = |x|$ if $X_i \subseteq \mathbb{R}$ and $||x|| = \max_{1 \leq n \leq m} |x_n|$ otherwise. The symbol $-i$ denotes "all players but i." In particular, $X_{-i} = \times_{j \neq i} X_j$.

We classify normal form games according to the properties of their pure strategy sets and payoff functions. A normal form game G is

1. *simple* if u_i is a simple function (i.e., a function with the property that its range is a finite set) for all $i \in N$;
2. *upper semicontinuous* if u_i is upper semicontinuous for all $i \in N$;
3. *convex* if X_i is convex for all $i \in N$;
4. *polyhedral* if X_i is polyhedral for all $i \in N$.

Note that in the last case, X_i is a polytope for all $i \in N$ (since X_i is compact and, hence, bounded).

For all games G and players $i \in N$, let $V_i : X_{-i} \to \mathbb{R}$ be defined by $V_i(x_{-i}) = \sup_{x_i \in X_i} u_i(x_i, x_{-i})$. A game G is *strongly upper semicontinuous* if G is upper semicontinuous and

$$\limsup_k V_i(\alpha_k z_{-i}^k + (1-\alpha_k)x_{-i}) \leq \limsup_k V_i(z_{-i}^k) \tag{1}$$

for all $i \in N$, $x_{-i} \in X_{-i}$, $\{z_{-i}^k\}_{k=1}^\infty$ converging to x_{-i} and $\{\alpha_k\}_{k=1}^\infty \subseteq (0,1]$ converging to zero. This notion was introduced by Ziad (1997) under the name of $i-$ upper semicontinuity.

A game G is *strongly quasi-concave* if, for all $i \in N$, there exists a finite convex compact cover $\{X_{-i}^l\}_{l=1}^{L_i}$ of X_{-i} such that u_i is quasi-concave on $X_i \times$

X_{-i}^l for all $l = 1, \ldots, L_i$. This notion was first introduced by Radzik (1991) and generalized by Ziad (1997).

A game G is *polyhedral strongly quasi-concave* if G is polyhedral and, for all $i \in N$, there exists a finite polyhedral cover $\{X_{-i}^l\}_{l=1}^{L_i}$ of X_{-i} such that u_i is quasi-concave on $X_i \times X_{-i}^l$ for all $l = 1, \ldots, L_i$. This notion differs from the above only in that X_{-i}^l is a polyhedral for all $i \in N$ and $l = 1, \ldots, L_i$ and not just compact and convex.

A game G is *strongly quasi-polyhedral concave* if G is polyhedral and, for all $i \in N$, there exists a finite polyhedral cover $\{X_{-i}^l\}_{l=1}^{L_i}$ of X_{-i} such that u_i is quasi-polyhedral-concave on $X_i \times X_{-i}^l$ for all $l = 1, \ldots, L_i$, i.e., the set

$$\{(x_i, x_{-i}) \in X_i \times X_{-i}^l : u_i(x_i, x_{-i}) \geq \alpha\} \tag{2}$$

is polyhedral for all $\alpha \in \mathbb{R}$. Alternatively, we could require that X_{-i} be covered by convex compact subsets. However, the apparently more general definition is equivalent to the above. Indeed, if $\alpha \leq \inf_{x \in X} u_i(x)$, then $X_i \times X_{-i}^l = \{x \in X_i \times X_{-i}^l : u_i(x) \geq \alpha\}$ is a polytope. Thus, if X_{-i}^l is convex, then, in fact, X_{-i}^l is a polytope.[3]

A game G is *best-reply strongly quasi-polyhedral concave* if G is polyhedral

[3] Since X is a polytope, there exists $\{a^1, \ldots, a^k\}$ such that $X = \text{co}(\{a^1, \ldots, a^k\})$. Then, $X_{-i} = \text{co}(\{a_{-i}^1, \ldots, a_{-i}^k\})$.

and, for all $i \in N$, there exists a finite polyhedral cover $\{X_{-i}^l\}_{l=1}^{L_i}$ of X_{-i} such that V_i is quasi-polyhedral-concave on X_{-i}^l for all $l = 1, \ldots, L_i$.

Given a game G and $\varepsilon \geq 0$, an ε – *equilibrium* of G is $x^* \in X$ such that

$$u_i(x^*) \geq u_i(x_i, x_{-i}^*) - \varepsilon \tag{3}$$

for all $i \in N$ and $x_i \in X_i$. A *Nash equilibrium* of G is an ε – equilibrium for $\varepsilon = 0$.

For all $\varepsilon > 0$ and $i \in N$, the *player i's* ε – *best-reply correspondence* is $BR_i^\varepsilon : X \rightrightarrows X_i$ defined by

$$BR_i^\varepsilon = \overline{\{x_i \in X_i : u_i(x) > \sup_{\hat{x}_i \in X_i} u_i(\hat{x}_i, x_{-i}) - \varepsilon\}}$$

for all $x \in X$.[4] The ε – *best-reply correspondence* is $BR_\varepsilon : X \rightrightarrows X$ defined by $BR_\varepsilon(x) = BR_1^\varepsilon(x) \times \cdots \times BR_n^\varepsilon(x)$ for all $x \in X$. *Player i's best-reply correspondence* is $BR_i : X \rightrightarrows X_i$ defined by

$$BR_i = \{x_i \in X_i : u_i(x) \geq \sup_{\hat{x}_i \in X_i} u_i(\hat{x}_i, x_{-i})\}$$

for all $x \in X$ and the *best-reply correspondence*, denoted by BR, equals the product of the individual best-reply correspondences.

[4]For all subsets A of a topological space Y, $\overline{A}$ denotes the closure of A.

3 Polyhedral Convexity and the Lower Hemicontinuity of the Best-Reply Correspondence

The critical aspect of Ziad's approach to the existence of approximate equilibria is to show that the approximate best-reply correspondence is lower hemicontinuous. In the particular case of simple games and letting $\varepsilon > 0$ be small,[5] this claim would imply that all simple, strongly upper semicontinuous and strongly quasi-concave games have a lower hemicontinuous best-reply correspondence.

In contrast to what is stated in Ziad (1997), we present an example of a simple, strongly upper semicontinuous and strongly quasi-concave game with a best-reply correspondence that fails to be lower hemicontinuous.

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = (x-1)^2$ and let

$$X_1 = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \text{ and } f(x) \leq y \leq x\}. \tag{4}$$

Since f is strictly convex and continuous, then X_1 is convex and compact.

[5] If for all $i \in N$, $u_i(X) = \{d_1^i, \ldots, d_{L_i}^i\}$, with $d_1^i < \ldots < d_{L_i}^i$ and if $0 < \varepsilon < \min_{i \in N} \min_{l \in \{1,\ldots,L_i - 1\}} (d_{l+1}^i - d_l^i)$, then the $\varepsilon -$ best-reply correspondence coincides with the best-reply correspondence.

Let $X_2 = [0, 1]$. Define

$$A = \{(x, y, z) \in \mathbb{R}^3 : x < 1, y = f(x) \text{ and } z = 0\}, \tag{5}$$

$B = \{(1, 0, z) \in \mathbb{R}^3 : z \in [0, 1]\}$ and $C = \text{co}(A \cup B)$. The payoff function for player 2 is

$$u_2(x, y, z) = \begin{cases} 1 & \text{if } (x, y, z) \in C, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Note that u_2 is quasi-concave in $X_1 \times X_2$. Furthermore, $A \cup B$, hence $\text{co}(A \cup B)$, is compact, and so u_2 is upper semicontinuous.

Define $u_1 \equiv 0$. Clearly, u_1 is quasi-concave and continuous. Finally, let $G = (X_1, X_2, u_1, u_2)$. Then, G is upper semicontinuous and strongly quasi-concave.

We claim that G is also strongly upper semicontinuous. It is clear that $V_1(x_2) = 0$ for all $x_2 \in X_2$. Since both $(1, 0, 0)$ and $(0, 1, 0)$ belong to $A \cup B$, it follows that $\{(x, y, z) : x = y \text{ and } z = 0\}$ is contained in $C = \text{co}(A \cup B)$. Hence, it is easy to see that $X_1 \times \{0\}$ is also contained in C. This implies that $V_2(x_1) = 1$ for all $x_1 \in X_1$, and the game is therefore strongly upper semicontinuous.

We finally show that the best-reply correspondence is not lower hemicontinuous. For that, it is enough to show that player 2's best-reply correspondence is not lower hemicontinuous.

Note that $C = \text{co}(A \cup B) = \cup_{\lambda\in[0,1]}(\lambda A + (1-\lambda)B)$ by Rockafellar (1970, Theorem 3.3, p. 18). This implies that if $(x, y, z) \in C$, $x < 1$ and $y = f(x)$, then $z = 0$, i.e., $(x, y, z) \in A$. Indeed, there exist $(\bar{x}, \bar{y}, \bar{z}) \in A$, $(\hat{x}, \hat{y}, \hat{z}) \in B$ and $\lambda \in [0, 1]$ such that $(x, y, z) = \lambda(\bar{x}, \bar{y}, \bar{z}) + (1-\lambda)(\hat{x}, \hat{y}, \hat{z})$. Note that $\hat{x} = 1$ and $\hat{y} = 0$; hence, $\hat{y} = f(\hat{x})$ and $\bar{x} \neq \hat{x}$. Since $x < 1$, then $\lambda > 0$. In order to get a contradiction, suppose that $\lambda < 1$. Then,

$$y = \lambda\bar{y} + (1-\lambda)\hat{y} = \lambda f(\bar{x}) + (1-\lambda)f(\hat{x}) > f(\lambda\bar{x} + (1-\lambda)\hat{x}) = f(x), \quad (7)$$

a contradiction. So, $\lambda = 1$ and $(x, y, z) = (\bar{x}, \bar{y}, \bar{z}) \in A$.

This fact implies that $BR_2(x, f(x)) = \{0\}$ for all $x \in [0, 1)$. Clearly, $BR_2(1, 0) = [0, 1]$. Hence, $1 \in BR_2(1, 0)$, $(1-1/k, f(1-1/k)) \to (1, 0)$ but there is no sequence $\{z_k\}_k$ such that $z_k \in BR_2(1-1/k, f(1-1/k))$ and $z_k \to 1$. Therefore, BR_2 is not lower hemicontinuous.

As will be shown below, the best-reply correspondence fails to be lower hemicontinuous because G is not polyhedral.

4 Characterizations of Polytopes and Ziad's Theorem

The main point of the paper is that polyhedral convexity is an important property to establish the lower hemicontinuity of the best-reply correspondence, and therefore, to establish the existence of Nash equilibria.

In order to prove our claim, we start by providing two useful characterizations of polytopes that will be used throughout the paper. Before we state our characterizations, recall that a polytope is the convex hull of finitely many points (see Rockafellar (1970, p. 12)) and let $E(C)$ denote the set of extreme points of a convex set C.

Proposition 1 *Let $P \subseteq \mathbb{R}^n$. Then, the following conditions are equivalent:*

1. *P is a polytope;*

2. *P is compact, convex and satisfies the following property: for all $x \in P$, there exists $r > 0$ such that*

$$\frac{r}{||\tilde{x} - x||}\tilde{x} + \left(1 - \frac{r}{||\tilde{x} - x||}\right) x \in P$$

for all $\tilde{x} \in P$, $\tilde{x} \neq x$;

3. P is compact, convex and satisfies the following property: for all $x \in P$, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \cap E(P) \subseteq \{x\}$.

Proof. We will prove that 1 implies 2, 2 implies 3 and 3 implies 1.

Let P be a polytope. Then, clearly, P is convex and compact. Suppose first that $P = \Delta_{n-1}$, where Δ_{n-1} denotes the standard $n-1$ – dimensional unit simplex in $\mathbb{R}^n$. Let $x \in \Delta_{n-1}$. Then, $x = (\lambda_1, \ldots, \lambda_n)$, where $\sum_{k=1}^n \lambda_k = 1$ and $\lambda_k \geq 0$ for all $k = 1, \ldots, n$. Define $r = \min\{\lambda_k : \lambda_k > 0\}$. Let $\tilde{x} \in \Delta_{n-1}$ be such that $x \neq \tilde{x}$ and let $\tilde{x} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n)$.

For convenience, let

$$\bar{x} = \frac{r}{||\tilde{x} - x||}\tilde{x} + \left(1 - \frac{r}{||\tilde{x} - x||}\right) x$$

and

$$\bar{\lambda}_k = \frac{r}{||\tilde{x} - x||}\tilde{\lambda}_k + \left(1 - \frac{r}{||\tilde{x} - x||}\right) \lambda_k$$

for all $k = 1, \ldots, n$. Then, $\bar{x} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_n)$. Clearly, $\sum_{k=1}^n \bar{\lambda}_k = 1$. Furthermore, for all $k = 1, \ldots, n$,

$$\bar{\lambda}_k = \lambda_k + r\frac{\tilde{\lambda}_k - \lambda_k}{\max_i |\tilde{\lambda}_i - \lambda_i|} \geq \begin{cases} 0 & \text{if } \lambda_k = 0, \\ \lambda_k - r \geq 0 & \text{if } \lambda_k > 0. \end{cases} \tag{8}$$

Hence, $\bar{x} \in \Delta_{n-1}$.

Finally, consider the general case. Since P is a polytope, let $a_1, \ldots, a_n \in \mathbb{R}^m$ be such that $P = \mathrm{co}(\{a_1, \ldots, a_n\})$. We may assume that $n \geq 2$ since otherwise there is nothing to prove.

Let $\pi : \mathbb{R}^n \to \mathbb{R}^m$ be the linear function defined by $\pi(e_i) = a_i$ for all $i = 1, \ldots, n$. One easily sees that $\pi(\Delta_{n-1}) = P$. Also, it follows that $||\pi|| > 0$ since $||\pi|| = \max_{||y||=1} ||\pi(y)|| \geq \max_{1 \leq i \leq n} ||\pi(e_i)|| = \max_{1 \leq i \leq n} ||a_i|| > 0$ since there are at least two points in $\{a_1, \ldots, a_n\}$.

Let $x \in P$ and let $y \in \Delta_{n-1}$ be such that $\pi(y) = x$. Then, there exists $\hat{r} > 0$ such that

$$\frac{\hat{r}}{||\tilde{y} - y||}\tilde{y} + \left(1 - \frac{\hat{r}}{||\tilde{y} - y||}\right) y \in \Delta_{n-1}$$

for all $\tilde{y} \in \Delta_{n-1}$, $\tilde{y} \neq y$.

Define $r = \hat{r}/||\pi||$ and let $\tilde{x} \in P$, $\tilde{x} \neq x$. Let $\tilde{y} \in \Delta_{n-1}$ be such that $\pi(\tilde{y}) = \tilde{x}$ and define

$$\bar{y} = \frac{\hat{r}}{||\tilde{y} - y||}\tilde{y} + \left(1 - \frac{\hat{r}}{||\tilde{y} - y||}\right) y \in \Delta_{n-1}.$$

Thus, $\pi(\bar{y}) \in P$, that is,

$$\bar{x} := \frac{\hat{r}}{||\pi(\tilde{x}) - \pi(x)||}\tilde{x} + \left(1 - \frac{\hat{r}}{||\pi(\tilde{x}) - \pi(x)||}\right) x \in P.$$

For convenience, let $\gamma = \hat{r}/||\pi(\tilde{x}) - \pi(x)||$ and $\alpha = \hat{r}/(||\pi||||\tilde{x} - x||) = r/||\tilde{x} - x||$. Since $||\pi(\tilde{x}) - \pi(x)|| \leq ||\pi||||\tilde{x} - x||$, then $\alpha \leq \gamma$. Thus, letting

$\theta = \alpha/\gamma \in (0, 1]$, it follows that $\alpha\tilde{x}+(1-\alpha)x = \theta(\gamma\tilde{x}+(1-\gamma)x)+(1-\theta)x \in P$ since both x and $\gamma\tilde{x} + (1-\gamma)x = \bar{x}$ belong to P. Thus,

$$\frac{r}{||\tilde{x} - x||}\tilde{x} + \left(1 - \frac{r}{||\tilde{x} - x||}\right) x = \alpha\tilde{x} + (1-\alpha)x \in P.$$

This proves 2.

Suppose that P satisfies 2. We claim that for all $x \in P$, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \cap E(P) \subseteq \{x\}$.

Let $x \in P$. It is enough to show that no sequence $\{x_k\}_{k=1}^{\infty}$ satisfying $x_k \neq x$ and $x_k \in E(P)$ for all $k \in \mathbb{N}$ converges to x.

Let $\{x_k\}_{k=1}^{\infty}$ be such that $x_k \neq x$ and $x_k \in E(P)$ for all $k \in \mathbb{N}$ and assume, in order to reach a contradiction, that x_k converges to x.

Let $\theta_k = 1/||x_k - x||^{1/2}$ and $\hat{x}_k = \theta_k x_k + (1-\theta_k)x$ for all $k \in \mathbb{N}$. Then, there exists $r > 0$ such that

$$\frac{r}{||x_k - x||}x_k + \left(1 - \frac{r}{||x_k - x||}\right) x \in P.$$

Since $||x_k - x|| \to 0$, there exists $K \in \mathbb{N}$ such that

$$\theta_k = \frac{1}{||x_k - x||^{1/2}} \leq \frac{r}{||x_k - x||}$$

for all $k \geq K$. Hence, for all $k \geq K$, $\hat{x}_k \in P$. Letting $\alpha_k = 1/\theta_k$, then $x_k = \alpha_k\hat{x}_k + (1-\alpha_k)x$ and $\alpha_k \in (0, 1)$ for all $k \geq K$. But this is a contradiction, since $x \in P$, $\hat{x}_k \in P$, $\hat{x}_k \neq x$, $\alpha_k \in (0,1)$ and x_k is an extreme point of P. This contradiction establishes 3.

Finally, we show that 3 implies 1. For all $x \in P$, let $\varepsilon(x) > 0$ be such that $B_{\varepsilon(x)}(x) \cap E(P) \subseteq \{x\}$. Then, $\{P \cap B_{\varepsilon(x)}(x)\}_{x \in P}$ is an open cover of P, and since P is compact, then there exists $\{x_1, \ldots, x_m\} \subseteq P$ such that $P = \cup_{j=1}^m B_{\varepsilon_j}(x_j) \cap P$, where $\varepsilon_j = \varepsilon(x_j)$ for all $j = 1, \ldots, m$. Then, it follows that

$$E(P) = E(P) \cap P = \cup_{j=1}^m B_{\varepsilon_j}(x_j) \cap E(P) \subseteq \cup_{j=1}^m \{x_j\}$$

and so $E(P) \subseteq \{x_1, \ldots, x_m\}$. Hence, P is a polytope. ■

The equivalence between the first and third properties simply states that a compact, convex set is a polytope if and only if its extreme points are isolated.

The second property is also easy to understand when $r \leq ||x - \tilde{x}||$, since in this case its conclusion follows readily from the convexity of P. The interesting case occurs when $||x - \tilde{x}||$ is smaller than r: in this case, the point $\theta\tilde{x} + (1-\theta)x$ with $\theta = r/||\tilde{x} - x|| > 1$ corresponds to connecting x and $\tilde{x}$ with a line and, starting from x, going beyond $\tilde{x}$. The equivalence between 1 and 2 shows that this can be done for all points $\tilde{x}$ and for some $r > 0$ in a way that the resulting point is still in P if and only if P is a polytope.

Proposition 1 allows us to understand easily why Ziad's proof fails and to correct his statement. In the course of his argument to show that BR_i^{ε} is lower hemicontinuous,[6] Ziad considers a sequence $\{x_{-i,k}\}_{k=1}^{\infty} \subseteq X_{-i}$ converging to a point x_{-i} and defines

$$z_{-i,k} = \frac{1}{||x_{-i,k} - x_{-i}||^{1/2}} x_{-i,k} + \left(1 - \frac{1}{||x_{-i,k} - x_{-i}||^{1/2}}\right) x_{-i}$$

for all $k \in \mathbb{N}$. If X_{-i} is a polytope, then it follows that $z_{-i,k}$ belongs to X_{-i} for all k sufficiently large. In fact, let $r > 0$ be given by property 2 and let $K \in \mathbb{K}$ be such that $k \geq K$ implies that $||x_{-i,k} - x_{-i}||^{1/2} \leq r$. Then, $1/||x_{-i,k} - x_{-i}||^{1/2} \leq r/||x_{-i,k} - x_{-i}||$, which implies that $z_{-i,k}$ can be expressed as a convex combination of x_{-i} and $\theta_k x_{-i,k} + (1 - \theta_k) x_{-i}$, with $\theta_k = r/||x_{-i,k} - x_{-i}||$. Since the latter point belongs to X_{-i}, then $z_{-i,k}$ also belongs to X_{-i}.

In contrast, as the example in the previous section shows, this conclusion may fail if X_{-i} is not a polytope. Hence, the correct statement of Ziad's Theorem is:

[6]There is also a mistake in the statement of Lemma 3.3, since it claims that BR_i^{ε} is upper hemicontinuous instead of lower hemicontinuous. This also implies that the existence of an approximate equilibrium is a consequence of Michael's selection theorem and Brouwer's fixed point theorem as in Radzik (1991) and not of Kakutani's fixed point theorem as claimed in Ziad (1997).

Theorem 1 (Ziad) *If G is a polyhedral, strongly upper semicontinuos and strongly quasi-concave game, then G has an ε – equilibrium for all $\varepsilon > 0$.*

5 A Characterization of Strong Upper Semicontinuity

As we have shown in the previous section, the polyhedral convexity of the action spaces is essential to the lower hemicontinuity of the (approximate) best-reply correspondence and, therefore, to any approach to the existence of (approximate) equilibria based on that property.

The importance of polyhedral convexity is strengthened here by relating the polyhedral concavity of the players' value functions to strong upper semicontinuity. In particular, we show that strong upper semicontinuity is equivalent to best-reply strong quasi-polyhedral concavity in simple, upper semicontinuous and polyhedral strongly quasi-concave games.

Proposition 2 *Let G be a simple, upper semicontinuous and polyhedral strongly quasi-concave game. Then, G is strongly upper semicontinuous if and only if G is best-reply strongly quasi-polyhedral concave.*

In order to prove Proposition 2, we start by establishing a result of independent interest since it shows that in upper semicontinuous games (not necessarily simple), best-reply strong quasi-polyhedral concavity is a sufficient condition for strong upper semicontinuity.

Lemma 1 *Let G be a best-reply strongly quasi-polyhedral concave game. Then,* $\limsup_k V_i(\alpha_k z_{-i}^k + (1-\alpha_k)x_{-i}) \leq \limsup_k V_i(z_{-i}^k)$ *for all* $i \in N$, $x_{-i} \in X_{-i}$, $\{z_{-i}^k\}_{k=1}^{\infty}$ *converging to* x_{-i} *and* $\{\alpha_k\}_{k=1}^{\infty} \subseteq (0,1]$ *converging to zero. Thus, if, in addition, G is upper semicontinuous, then G is strongly upper semicontinuous.*

Proof. Let G be a normal form game and assume that G is best-reply strongly quasi-polyhedral concave.

Let $x_{-i} \in X_{-i}$, $\{x_{-i}^k\}_{k=1}^{\infty} \subseteq X_{-i}$ be a sequence converging to x_{-i} and $\{\alpha_k\}_{k=1}^{\infty} \subseteq (0,1]$ converging to zero. Let $\bar{x}_{-i}^k = \alpha_k x_{-i}^k + (1-\alpha_k)x_{-i}$ for all $k \in \mathbb{N}$. We may assume that $x_{-i}^k \neq x_{-i}$ for infinitely many k's, since otherwise $\limsup_k V_i(x_{-i}^k) = \limsup_k V_i(\bar{x}_{-i}^k) = V_i(x_{-i})$. Hence, taking a subsequence if necessary, we may assume that $x_{-i}^k \neq x_{-i}$ for all $k \in \mathbb{N}$; clearly, this implies that $\bar{x}_{-i}^k \neq x_{-i}$ for all $k \in \mathbb{N}$.

Let $\gamma = \limsup_k V_i(\bar{x}_{-i}^k)$. Since V_i is simple, there exists a subsequence $\{\bar{x}_{-i}^{k_j}\}_j$ of $\{\bar{x}_{-i}^k\}_k$ such that $\lim_{j\to\infty} V_i(\bar{x}_{-i}^{k_j}) = \gamma$. Let $\varepsilon > 0$ and let $J_1 \in \mathbb{N}$ be such that $V_i(\bar{x}_{-i}^{k_j}) > \gamma - \varepsilon$ for all $j \geq J_1$.

Let $\{X^l_{-i}\}_{l=1}^{L_i}$ be a polyhedral cover of X_{-i} such that V_i is quasi-polyhedral concave on X^l_{-i} for all $l = 1, \ldots, L_i$. Since the cover is finite, we may assume that there exists $l \in \{1, \ldots, L_i\}$ such that $\bar{x}^{k_j}_{-i} \in X^l_{-i}$ for all $j \in \mathbb{N}$. Since X^l_{-i} is compact, then $x_{-i} \in X^l_{-i}$. Letting $P = \{y \in X^l_{-i} : V_i(y) \geq \gamma - \varepsilon\}$, then P is a polytope and $\bar{x}^{k_j}_{-i} \in P$ for all $j \geq J_1$. Furthermore, $x_{-i} \in P$ since P is compact.

Since P is a polytope, let $r > 0$ be such that

$$\frac{r}{||\tilde{x}_{-i} - x_{-i}||}\tilde{x}_{-i} + \left(1 - \frac{r}{||\tilde{x}_{-i} - x_{-i}||}\right) x_{-i} \in P$$

for all $\tilde{x}_{-i} \in P$, $\tilde{x}_{-i} \neq x_{-i}$ (see Proposition 1). Also, let $J_2 \in \mathbb{N}$ be such that $||x^{k_j}_{-i} - x_{-i}|| < r$ for all $j \geq J_2$.

Let $J = \max\{J_1, J_2\}$ and $j \geq J$. Then,

$$\hat{x}^{k_j}_{-i} := \frac{r}{||\bar{x}^{k_j}_{-i} - x_{-i}||}\bar{x}^{k_j}_{-i} + \left(1 - \frac{r}{||\bar{x}^{k_j}_{-i} - x_{-i}||}\right) x_{-i} \in P.$$

Letting $\theta_{k_j} = r\alpha_{k_j}/(||\bar{x}^{k_j}_{-i} - x_{-i}||)$, it follows that $\hat{x}^{k_j}_{-i} = \theta_{k_j} x^{k_j}_{-i} + (1 - \theta_{k_j})x_{-i}$ and so

$$x^{k_j}_{-i} = \frac{1}{\theta_{k_j}}\hat{x}^{k_j}_{-i} + \left(1 - \frac{1}{\theta_{k_j}}\right) x_{-i}.$$

Since $||\bar{x}^{k_j}_{-i} - x_{-i}|| = \alpha_{k_j}||x^{k_j}_{-i} - x_{-i}||$ then $\theta_{k_j} = r/||x^{k_j}_{-i} - x_{-i}||$ and so $1/\theta_{k_j} \in (0, 1)$ because $||x^{k_j}_{-i} - x_{-i}|| < r$. Hence, it follows that $x^{k_j}_{-i} \in P$, which implies that $V_i(x^{k_j}_{-i}) \geq \gamma - \varepsilon$. Since this holds for all $j \geq J$, it follows that

$\limsup_k V_i(x^k_{-i}) \geq \gamma - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that

$$\limsup_k V_i(x^k_{-i}) \geq \gamma = \limsup_k V_i(\bar{x}^k_{-i})$$

and so u_i is strongly upper semicontinuous. ■

It is worth noting that the converse of Lemma 1 is false. For example, let G be defined by $X_1 = [0,1] \times [0,1]$, $X_2 = [0,1]$, $u_1(x,y,z) = 0$ and $u_2(x,y,z) = x + \sqrt{y}$ for all $(x,y) \in X_1$ and $z \in X_2$. Clearly, G is strongly upper semicontinuous since u_i is continuous for all $i = 1,2$. However, G is not best-reply strongly quasi-polyhedral concave: in order to reach a contradiction, suppose that $\{X_1^l\}_{l=1}^L$ is a polyhedral cover of X_1 such that $V_2 = u_2$ is quasi-polyhedral concave in X_1^l for all $l = 1,\ldots,L$. Let $(\hat{x},\hat{y})$ be such that it belongs to the interior of some X_1^l and let $\alpha = V_2(\hat{x},\hat{y})$. Then, $C := \{(x,y) \in X_1^l : V_2(x,y) \geq \alpha\} = X_1^l \cap \{(x,y) \in X_1 : x \geq \alpha - \sqrt{y}\}$, and so there is some $\varepsilon > 0$ such that $\{(x,y) \in X_1 : x = \alpha - \sqrt{y}\} \cap B_\varepsilon(\hat{x},\hat{y}) \subseteq C$. Points in $\{(x,y) \in X_1 : x = \alpha - \sqrt{y}\} \cap B_\varepsilon(\hat{x},\hat{y})$ are extreme points of C and so C is not a polytope. Hence, V_2 is not quasi-polyhedral concave in X_1^l, a contradiction.

We next turn to the remaining part of Proposition 2. The following lemma asserts that the quasi-(polyhedral) concavity of u_i is inherited by V_i.

Lemma 2 *Let $i \in N$, C_{-i} be a polyhedral subset of X_{-i} and u_i be upper semicontinuous. If u_i is quasi-concave in $X_i \times C_{-i}$, then V_i is quasi-concave in C_{-i}. Furthermore, if u_i is quasi-polyhedral concave in $X_i \times C_{-i}$, then V_i is quasi-polyhedral concave in C_{-i}.*

Proof. Let $\gamma \in \mathbb{R}$ and $\pi_{-i} : \mathbb{R}^m \times \mathbb{R}^{(n-1)m} \to \mathbb{R}^{(n-1)m}$ be the projection onto $\mathbb{R}^{(n-1)m}$. Then, $\{x_{-i} \in C_{-i} : V_i(x_{-i}) \geq \gamma\} = \pi_{-i}(\{x \in X_i \times C_{-i} : u_i(x) \geq \gamma\})$. In order to prove this claim, let $A = \{x_{-i} \in C_{-i} : V_i(x_{-i}) \geq \gamma\}$ and $B = \{x \in X_i \times C_{-i} : u_i(x) \geq \gamma\}$. If $x_{-i} \in A$, then $V_i(x_{-i}) \geq \gamma$. Since u_i is upper semicontinuous, there exists $x_i \in X_i$ such that $u_i(x_i, x_{-i}) = V_i(x_{-i}) \geq \gamma$. Hence, $(x_i, x_{-i}) \in B$ and $x_{-i} = \pi_{-i}(x_i, x_{-i})$, implying that $x_{-i} \in \pi_{-i}(B)$. Conversely, if $x_{-i} \in \pi_{-i}(B)$, then there exists $x_i \in X_i$ such that $x_{-i} = \pi_{-i}(x_i, x_{-i})$ and $(x_i, x_{-i}) \in B$. Since $V_i(x_{-i}) \geq u_i(x_i, x_{-i}) \geq \gamma$, it follows that $x_{-i} \in A$.

If u_i is quasi-concave in $X_i \times C_{-i}$, then $\{x \in X_i \times C_{-i} : u_i(x) \geq \gamma\}$ is convex and since π_{-i} is linear then, by Rockafellar (1970, Theorem 19.3, p. 174), $\{x_{-i} \in C_{-i} : V_i(x_{-i}) \geq \gamma\}$ is convex.

Similarly, if u_i is quasi-polyhedral concave in $X_i \times C_{-i}$, then $\{x \in X_i \times C_{-i} : u_i(x) \geq \gamma\}$ is polyhedral, and so $\{x_{-i} \in C_{-i} : V_i(x_{-i}) \geq \gamma\}$ is also polyhedral by Rockafellar (1970, Theorem 19.3, p. 174). ■

The following lemma shows that if players' value functions are polyhedral strongly quasi-concave in a simple, strong upper semicontinuous game, then they are in fact strongly quasi-polyhedral concave.

Lemma 3 *If G is simple, strong upper semicontinuous and polyhedral strongly quasi-concave, then, G is best-reply strongly quasi-polyhedral concave.*

Proof. Let G be a simple and strongly upper semicontinuous game and, for all $i \in N$, let $\{X^l_{-i}\}_{l=1}^{L_i}$ be a polyhedral cover of X_{-i} such that u_i is quasi concave in $X_i \times X^l_{-i}$ for all $l = 1, \ldots, L_i$. By Lemma 2, it follows that V_i is quasi-concave in X^l_{-i} for all $l = 1, \ldots, L_i$.

Let $l \in \{1, \ldots, L_i\}$. We will show that V_i is quasi-polyhedral concave in X^l_{-i}. Let $\gamma \in \mathbb{R}$ and define $P = \{x_{-i} \in X^l_{-i} : V_i(x_{-i}) \geq \gamma\}$. Since u_i is upper semicontinuous, V_i is upper semicontinuous (see Berge (1997, Theorem 2, p.116)), and so P is compact. Since V_i is quasi-concave in X^l_{-i}, then P is convex. In order to show that P is a polytope, it is enough to show that for all $x_{-i} \in P$, there exists $\varepsilon > 0$ such that $B_\varepsilon(x_{-i}) \cap E(P) \subseteq \{x_{-i}\}$ (see Proposition 1).

Let $x_{-i} \in P$. In order to prove the above claim, it suffices to show that no sequence $\{x^k_{-i}\}_{k=1}^{\infty}$ satisfying $x^k_{-i} \neq x_{-i}$ and $x^k_{-i} \in E(P)$ for all $k \in \mathbb{N}$ converges to x_{-i}.

Let $\{x_{-i}^k\}_{k=1}^{\infty}$ be such that $x_{-i}^k \neq x_{-i}$ and $x_{-i}^k \in E(P)$ for all $k \in \mathbb{N}$ and assume, in order to reach a contradiction, that x_{-i}^k converges to x_{-i}.

Let $\theta_k = 1/||x_{-i}^k - x_{-i}||^{1/2}$ and $\hat{x}_{-i}^k = \theta_k x_{-i}^k + (1-\theta_k)x_{-i}$ for all $k \in \mathbb{N}$. Since X_{-i}^l is a polytope, by Proposition 1, there exists $r > 0$ such that

$$\frac{r}{||x_{-i}^k - x_{-i}||} x_{-i}^k + \left(1 - \frac{r}{||x_{-i}^k - x_{-i}||}\right) x_{-i} \in X_{-i}^l.$$

Since $||x_{-i}^k - x_{-i}|| \to 0$, there exists $K \in \mathbb{N}$ such that

$$\theta_k = \frac{1}{||x_{-i}^k - x_{-i}||^{1/2}} \leq \frac{r}{||x_{-i}^k - x_{-i}||}$$

for all $k \geq K$. Hence, for all $k \geq K$, $\hat{x}_{-i}^k \in X_{-i}^l$.

Furthermore, $||\hat{x}_{-i}^k - x_{-i}|| = \theta_k ||x_{-i}^k - x_{-i}|| = ||x_{-i}^k - x_{-i}||^{1/2}$ and so $\hat{x}_{-i}^k$ converges to x_{-i}. Letting $\alpha_k = 1/\theta_k$, then $x_{-i}^k = \alpha_k \hat{x}_{-i}^k + (1-\alpha_k)x_{-i}$ and $\alpha_k \in (0,1)$ for all $k \geq K$.

Since u_i is strongly upper semicontinuous, then

$$\limsup_k V_i(\hat{x}_{-i}^k) \geq \limsup_k V_i(x_{-i}^k) \geq \gamma.$$

Since u_i is simple, there exists $\bar{k} \geq K$ such that $V_i(\hat{x}_{-i}^{\bar{k}}) \geq \gamma$, i.e., $\hat{x}_{-i}^{\bar{k}} \in P$. But this is a contradiction, since $x_{-i} \in P$, $x_{-i}^{\bar{k}} = \alpha_{\bar{k}} \hat{x}_{-i}^{\bar{k}} + (1-\alpha_{\bar{k}})x_{-i}$, $\hat{x}_{-i}^{\bar{k}} \neq x_{-i}$, $\alpha_{\bar{k}} \in (0,1)$ and $x_{-i}^{\bar{k}}$ is an extreme point of P. This contradiction proves the claim and the lemma follows. ■

Combining Lemma 2 and Lemma 3, we obtain the remaining part of Proposition 2, stated in the following corollary.

Corollary 1 *If G is a simple, strong upper semicontinuous, polyhedral strongly quasi-concave game, then G is best-reply strongly quasi-polyhedral concave.*

6 Relation between Radzik's and Ziad's Theorems

Proposition 2 is also useful to compare Ziad's theorem with Radzik's. Clearly, Ziad allows for more generality on the number of players and on the action spaces (since the action spaces in Radzik (1991) are intervals in $\mathbb{R}$, and these sets are polytopes). However, it might seem that Radzik's theorem allows for more general payoff functions since it only requires upper semicontinuity, but not strong upper semicontinuity. However, for two-player simple games on a square, strong upper semicontinuity is equivalent to upper semicontinuity. Therefore, Radzik's theorem is a corollary of Ziad's theorem 1.

Proposition 3 *If G is an upper semicontinuous and strongly quasi-concave two-player game, and X_i is a compact interval on the real line for all $i = 1, 2$,* *then G is best-reply strongly quasi-polyhedral concave. Hence, G is strongly upper semicontinuous.*

Proof. Let G be an upper semicontinuous and strongly quasi-concave game. Let $i \in N$. Obviously, X_i is polyhedral.

Let $\{X_{-i}^l\}_{l=1}^{L_i}$ be a compact convex partition of X_{-i} such that u_i is quasi-concave in $X_i \times X_{-i}^l$ for all $l = 1, \dots, L_i$. Again, X_{-i}^l is polyhedral for all l and so, in fact, $\{X_{-i}^l\}_{l=1}^{L_i}$ is a polyhedral partition of X_{-i}.

We claim that V_i is quasi-polyhedral concave in X_{-i}^l for all $l = 1, \dots, L_i$. Let $\alpha \in \mathbb{R}$. By Lemma 2, $\{x_{-i} \in X_{-i}^l : V_i(x_{-i}) \geq \alpha\}$ is a convex subset of $X_{-i} \subseteq \mathbb{R}$, and so an interval. Since G is upper semicontinuous, then $\{x_{-i} \in X_{-i}^l : V_i(x_{-i}) \geq \alpha\}$ is also closed. Hence, it is a closed interval and so a polytope. ■

Combining the above proposition with Ziad's Theorem, we obtain the main result in Radzik (1991).

Corollary 2 (Radzik) *If G is an upper semicontinuous and strongly quasi-concave two-player game, and X_i is a compact interval on the real line for all $i = 1, 2$, then G has an ε – equilibrium for all $\varepsilon > 0$.*

We can also obtain an existence result for n – person games played in polytopes that parallels the statement of Radzik's Theorem and is, again, a corollary of Ziad's.

Corollary 3 *If G is a strongly upper semicontinuous and best-reply strongly quasi-polyhedral concave game, then G has an ε – equilibrium for all $\varepsilon > 0$.*

In particular, if G is a strongly upper semicontinuous and strongly quasi-polyhedral concave game, then G has an ε – equilibrium for all $\varepsilon > 0$.

The first part of the above corollary follows at once from Ziad's Theorem and Lemma 1, while the second follows from the first and Lemma 2. Furthermore, using again Proposition 3, it follows that Corollary 3 implies Radzik's Theorem.

7 Concluding Remarks

The approach for the existence results discussed in this paper relies on the lower hemicontinuity of the best-reply correspondence. As the example in Section 3 shows, this condition may fail even if the game satisfies the assumptions in Ziad (1997), leading to the conclusion that the action spaces need to be polytopes in order for his result to hold.

Unfortunately, we have been unable to find an example of a game satisfying all of Ziad's assumptions but without $\varepsilon -$ equilibria for some $\varepsilon > 0$. In particular, note that in the example in Section 3, despite the fact that BR_2 is not lower hemicontinuous, it admits a continuous selection: $f(x_1) = 0$ for all $x_1 \in X_1$. Since the same is true for BR_1 we conclude that this game has a Nash equilibrium. Thus, two questions arise: Does the best-reply correspondence admit a continuous selection in all upper semicontinuous and strongly quasi-concave games even if players' action spaces are not polytopes? Does all upper semicontinuous and strongly quasi-concave games have a Nash equilibrium even if players' action spaces are not polytopes?

Regarding the characterization in Proposition 2, it seems possible to extend it from polyhedral strong quasi-concave games to strong quasi-concave games. A possible way to achieve such an extension is by proving the following intuitive conjecture: If $X \subseteq \mathbb{R}^m$ is a polytope and $\{X_1, \ldots, X_k\}$ is a convex, compact cover of X, does there exist a polyhedral cover $\{Y_1, \ldots, Y_q\}$ such that for all $j \in \{1, \ldots, q\}$ there exists $l \in \{1, \ldots, k\}$ such that $Y_j \subseteq X_l$? If this conjecture is true, then strong upper semicontinuity is equivalent to best-reply quasi-polyhedral concavity in all simple, upper semicontinuous and strongly quasi-concave games.

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